

COUNTEREXAMPLES TO C^∞ WELL POSEDNESS FOR SOME HYPERBOLIC OPERATORS WITH TRIPLE CHARACTERISTICS

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ABSTRACT. In this paper we prove that for a class of non-effectively hyperbolic operators with smooth triple characteristics the Cauchy problem is well posed in the Gevrey 2 class, beyond the generic Gevrey class $3/2$ (see e.g. [4]). Moreover we show that this value is optimal.

1. INTRODUCTION

Hyperbolic operators with double characteristics have been thoroughly investigated in the past years, and at least in the case when there is no transition between different types on the set where the principal symbol vanishes of order 2, essentially everything is known, see e.g. [14] and [2] for a general survey and [7] and [5] for classical introductions. The algebraic classification of the spectrum of the fundamental matrix of the principal symbol evaluated at a double point allows us to deduce the behavior of the operator in the C^∞ and Gevrey categories as far as the well posedness of the Cauchy problem is concerned. In particular, when real eigenvalues exist, the so called effectively hyperbolic case, then we have well posedness regardless of the lower order terms.

These spectral invariants are not available in general when studying operators with symbols vanishing of order greater or equal to 3, therefore much less is known in this case. There is one object, though, that allows some classification even in these cases, namely the propagation cone of the principal symbol, i.e. the symplectic dual of the hyperbolicity cone. More precisely we recall that, denoting by p_z the localization of the principal symbol p of $P(x, D)$ at a multiple point z , the propagation cone C_z is defined by

$$C_z = \{X \in T_z(T^*\mathbb{R}^{n+1}) \mid \sigma(X, Y) \leq 0, \forall Y \in \Gamma_z\},$$

where the hyperbolicity cone Γ_z is defined as the connected component of $N = (0; 1, \dots, 0)$ of the set $\{X \in T_z(T^*\mathbb{R}^{n+1}) \mid p_z(X) \neq 0\}$, assuming that $p(x, \xi)$ is hyperbolic with respect to ξ_0 .

When C_z happens to be transversal to the tangent plane to the manifold of multiple points, we are again effectively hyperbolic as it were, i.e. if

characteristics are double, it can be shown that this is equivalent to the spectrum of the fundamental matrix containing real eigenvalues ([12], [5]). When this happens in a higher order multiplicity situation and the lower order terms satisfy a generic Ivrii-Petkov vanishing condition, it is known that we have well posedness in C^∞ . See [9] for a very complete analysis of this situation and [3] for some new recent results in triple characteristics of an effectively hyperbolic type. One strongly suspects that when this transversality condition fails, it may be always possible to choose some suitable lower order terms satisfying Ivrii-Petkov conditions and still end up with an ill posed problem in C^∞ .

At least in the case of triple characteristics this behavior has been proved in a number of papers, see e.g. [1], [8], [12], but the principal symbol had to satisfy some strong factorization conditions, where one or all of the roots had to be C^∞ .

In this paper we prove a well posedness result in the Gevrey category for a simple model hyperbolic operator with triple characteristics, when however there are no regular roots, i.e. the principal symbol cannot be smoothly factorized, and moreover whose propagation cone is not transversal to the triple manifold, thus confirming that conjecture, albeit for a limited class of operators. On the other hand here we are able not only to disprove C^∞ well posedness, but we can actually estimate the precise Gevrey threshold where well posedness will cease to hold, by exhibiting a special class of solutions, through which we can violate weak necessary solvability conditions. This threshold will appear at $s = 2$, thus beyond the canonical value of $s = \frac{3}{2}$ dictated by the classical result of Bronshtein, [4]. The choice of the lower order terms will be the easiest possible, i.e. zero. It is thus all the more surprising that a very regular operator, with analytic (polynomial) coefficients, and reduced just to its principal symbol should have this bad behavior, with respect to C^∞ well posedness.

We consider the operator

$$(1) \quad P(x, D) = D_0^3 - \Omega(x, D')D_0 + B(x, D')$$

with

$$\Omega(x, D') = D_1^2 + x_1^2 D_n^2$$

and $B(x, D') = -b_0 x_1^3 D_n^3$.

Here $x = (x_0, x_1, \dots, x_n) = (x_0, x') = (x_0, x_1, x'', x_n) \in \mathbb{R}^{n+1}$ and the local estimates below will be proven in a neighborhood of $x = 0$. Clearly hyperbolicity is equivalent to $b_0^2 \leq \frac{4}{27}$. We will also assume that the

principal symbol vanishes exactly of order 3 on the triple manifold Σ_3 , thus we will require $|b_0| < \frac{2}{3\sqrt{3}}$, i.e. outside Σ_3 P is strictly hyperbolic.

Let us recall that we say that $f(x) \in C^\infty(\mathbb{R}^n)$ belongs to $\gamma^{(s)}(\mathbb{R}^n)$, the Gevrey space of order s , where $s \geq 1$, if for any compact set $K \subset \mathbb{R}^n$ there exist $C > 0$, $h > 0$ such that

$$(2) \quad |\partial_x^\alpha f(x)| \leq Ch^{|\alpha|} |\alpha|!^s, \quad x \in K$$

for every $\alpha \in \mathbb{N}^n$. In particular $\gamma^{(1)}(\mathbb{R}^n)$ is the space of real analytic functions on \mathbb{R}^n .

We also recall that the Cauchy problem for P is said to be locally solvable in $\gamma^{(s)}$ at the origin if for any $\Phi = (u_0, u_1) \in (\gamma^{(s)}(\mathbb{R}^n))^2$, there exists a neighborhood U_Φ of the origin such that the Cauchy problem

$$\begin{cases} Pu = 0 & \text{in } U_\Phi \\ D_0^j u(0, x') = u_j(x'), & j = 0, 1, \quad x \in U_\Phi \cap \{x_0 = 0\} \end{cases}$$

has a solution $u(x) \in C^\infty(U_\Phi)$.

The main results in this paper are then precisely stated:

Theorem 1.1. *Assume that $0 < |b_0| < \frac{2}{3\sqrt{3}}$. Then the Cauchy problem for P is well posed in the Gevrey 2 class.*

That this is actually the best one can hope for is proven in

Theorem 1.2. *If $s > 2$, it is possible to choose $b_0 \in]0, \frac{2}{3\sqrt{3}}[$ such that the Cauchy problem for P is not locally solvable at the origin in the Gevrey s class.*

The paper is organized as follows: in Section 2 we prove a simple, classical energy estimate for our model operator in the Gevrey s category, with $s \leq 2$ which proves Theorem 1.1.

In Section 3 we recall a number of results from [15], [13] and [2] on the entire functions related to a Stokes phenomenon for an important ODE associated with the necessary conditions. We finally prove Theorem 1.2 via a standard functional analytic argument, involving exponential estimates. Eventually in Section 4 we verify that the geometrical conditions on the propagation cone and the regularity of the roots for the principal symbol of our model hold true.

2. ESTIMATES IN GEVREY CLASSES

We define $\langle u, v \rangle = \int_{\mathbb{R}^n} \hat{u}(x_0, x_1, x', \xi_n) \overline{\hat{v}}(x_0, x_1, x', \xi_n) dx_1 dx'$ with \hat{u} denoting the partial Fourier transform with respect to x_n . In a similar way we have for the L^2 norm $\|u\|^2 = \int_{\mathbb{R}^n} |\hat{u}(x_0, x_1, x', \xi_n)|^2 dx_1 dx'$.

Since we are dealing a rather simple and straightforward model operator we are not going to deploy the techniques used e.g. in [2] of Weyl-Gevrey calculus of pseudo-differential operators. We could certainly apply them here, but at the price of rendering the computations very heavy, and then for a very limited advantage in generality.

Therefore the symbol $W(x_0) = \exp(2\tau\langle\xi_n\rangle^{\frac{1}{s}}(x_0 - a))$ with some $a > 0$ defined below will function as a Gevrey weight in a naive, still correct, way.

The harmonic oscillator Ω is defined as $\Omega = D_1^2 + x_1^2\xi_n^2$.

Before dealing with the operator (1) itself, we need a preliminary result on the multiplier operator M . Let

$$E_j(u(x_0)) = \|D_0^j u(x_0)\|^2 + \|D_1^j u(x_0)\|^2 + \|(x_1 \xi_n)^j u(x_0)\|^2.$$

Assuming that $\theta > 0$ we start by proving the following

Lemma 2.1. *Let $M = D_0^2 - \theta\Omega$. Then there exists $C > 0$ such that for any $s \geq 1, s \in \mathbb{R}$ and any τ large enough we have for any $u \in C_0^\infty(\mathbb{R}^n)$*

$$(3) \quad \begin{aligned} \int_0^\infty W \|Mu\|^2 dx_0 &\geq CW(0) (\tau\langle\xi_n\rangle^{\frac{1}{s}} E_1(u(0)) + \tau^3\langle\xi_n\rangle^{\frac{3}{s}} E_0(u(0))) \\ &+ C\tau^2 \int_0^\infty W \langle\xi_n\rangle^{\frac{2}{s}} E_1 u(x_0) dx_0 + C\tau^4 \int_0^\infty W \langle\xi_n\rangle^{\frac{4}{s}} E_0(u(x_0)) dx_0, \end{aligned}$$

where $W = \exp(2\tau\langle\xi_n\rangle^{\frac{1}{s}}(x_0 - a))$ and $\langle\xi_n\rangle = \sqrt{1 + \xi_n^2}$.

Proof. We compute

$$(4) \quad \begin{aligned} 2i \operatorname{Im}\langle Mu, D_0 u \rangle &= 2i \operatorname{Im}\langle D_0^2 u, D_0 u \rangle - \theta 2i \operatorname{Im}\langle \Omega u, D_0 u \rangle \\ &= D_0 \{ \|D_0 u\|^2 + \theta \langle \Omega u, u \rangle \} \\ &= D_0 E(u(x_0)). \end{aligned}$$

Therefore we have

$$(5) \quad \begin{aligned} 2 \int_0^\infty W \operatorname{Im}\langle Mu, D_0 u \rangle dx_0 &= W(0) E(u(0)) \\ &+ 2\tau \int_0^\infty W E(u(x_0)) \langle\xi_n\rangle^{\frac{1}{s}} dx_0. \end{aligned}$$

Since $\langle \Omega u, u \rangle = \|D_1 u\|^2 + \|x_1 \xi_n u\|^2$ we have

$$(6) \quad \begin{aligned} & 2 \int_0^\infty W \operatorname{Im} \langle Mu, D_0 u \rangle dx_0 = W(0)E(u(0)) \\ & + 2\tau \int_0^\infty W \langle \xi_n \rangle^{\frac{1}{s}} \{ \|D_0 u\|^2 + \theta \|D_1 u\|^2 + \theta \xi_n \|x_1 \xi_n u\|^2 \} dx_0. \end{aligned}$$

Using Cauchy-Schwarz inequality we see

$$(7) \quad \begin{aligned} & \int_0^\infty W \|Mu\|^2 \geq \tau \langle \xi_n \rangle^{\frac{1}{s}} W(0)E(u(0)) \\ & + \tau^2 \int_0^\infty W \langle \xi_n \rangle^{\frac{2}{s}} (\|D_0 u\|^2 + \theta \|D_1 u\|^2 + \theta \|x_1 \xi_n u\|^2) dx_0. \end{aligned}$$

Repeating similar arguments we have

$$\int_0^\infty W \|D_0 u\|^2 dx_0 \geq \tau \langle \xi_n \rangle^{\frac{1}{s}} W(0) \|u(0)\|^2 + \tau^2 \int_0^\infty W \langle \xi_n \rangle^{\frac{2}{s}} \|u\|^2 dx_0$$

and replacing $(\int_0^\infty W \|D_0 u\|^2 dx_0)/2$ by the above estimate the right-hand side of (7) is bounded from below by

$$\begin{aligned} & W(0) \left(\tau \langle \xi_n \rangle^{\frac{1}{s}} E_1(u(0)) + \frac{1}{2} \tau^2 \langle \xi_n \rangle^{\frac{3}{s}} E_0(u(0)) \right) \\ & + \tau^2 \int_0^\infty W \langle \xi_n \rangle^{\frac{2}{s}} (E_1(u(x_0)) + \tau^2 \langle \xi_n \rangle^{\frac{2}{s}} E_0(u(x_0))) dx_0. \end{aligned}$$

It is easy to see that (3) holds. \square

We now move to the proof of Theorem 1.1.

Proof. First notice that if $b_0 = 0$ the result is a trivial consequence of the double characteristics theory, and in that case we do have C^∞ well posedness, as it will also become clear from the estimates below. That is why we will assume that $b_0 \neq 0$. We will make use of standard energy estimates.

We choose $\theta = \frac{1}{3}$ and with $M(x, D) = D_0^2 - \frac{\Omega}{3}$ compute

$$(8) \quad \begin{aligned} & 2i \operatorname{Im} \langle Pu, Mu \rangle = 2i \operatorname{Im} \left\langle \left(D_0 M - \frac{2}{3} \Omega D_0 - B \right) u, Mu \right\rangle \\ & = D_0 \left\{ \|Mu\|^2 \right\} + 2i \operatorname{Im} \left\langle -\frac{2}{3} \Omega D_0 u, D_0^2 u \right\rangle + 2i \operatorname{Im} \left\langle \frac{2}{3} \Omega D_0 u, \frac{\Omega}{3} u \right\rangle \\ & \quad + 2i \operatorname{Im} \langle -b_0 x_1^3 \xi_n^3 u, D_0^2 u \rangle + 2i \operatorname{Im} \left\langle -b_0 x_1^3 \xi_n^3 u, -\frac{\Omega}{3} u \right\rangle. \end{aligned}$$

From (8) we get

$$(9) \quad 2i \operatorname{Im} \langle Pu, Mu \rangle = D_0 \mathcal{E}(u) + \mathcal{R}(u),$$

where $\mathcal{R}(u) = \frac{b_0}{3} \langle [D_1^2, x_1^3] \xi_n^3 u, u \rangle$ and

$$(10) \quad \mathcal{E}(u) = \|Mu\|^2 + \frac{2}{3} \langle \Omega D_0 u, D_0 u \rangle + \frac{2}{9} \|\Omega u\|^2 + 2b_0 \operatorname{Re} \langle x_1^3 \xi_n^3 u, D_0 u \rangle.$$

From (10) we have

$$(11) \quad \begin{aligned} \mathcal{E}(u) &= \|Mu\|^2 + 2b_0 \operatorname{Re} \langle x_1^2 \xi_n^2 u, x_1 \xi_n D_0 u \rangle \\ &\quad + \frac{2}{3} \left(\|D_1 D_0 u\|^2 + \|x_1 \xi_n D_0 u\|^2 \right) \\ &\quad + \frac{2}{9} \left(\|D_1^2 u\|^2 + \|x_1^2 \xi_n^2 u\|^2 + 2 \operatorname{Re} \langle D_1^2 u, x_1^2 \xi_n^2 u \rangle \right). \end{aligned}$$

We write (11) like this:

$$(12) \quad \begin{aligned} \mathcal{E}(u) &= \|Mu\|^2 + \frac{2}{3} \|D_1 D_0 u\|^2 \\ &\quad + \left\| \sqrt{\frac{2}{3}} x_1 \xi_n D_0 u + b_0 \sqrt{\frac{3}{2}} x_1^2 \xi_n^2 u \right\|^2 + \frac{2}{9} \|D_1^2 u\|^2 \\ &\quad + \frac{2}{9} \left(1 - \frac{27}{4} b_0^2 \right) \|x_1^2 \xi_n^2 u\|^2 + \frac{4}{9} \operatorname{Re} \langle D_1^2 u, x_1^2 \xi_n^2 u \rangle. \end{aligned}$$

Noticing that $\operatorname{Re} \langle D_1^2 u, x_1^2 u \rangle = \|x_1 D_1 u\|^2 - \|u\|^2$ we get from (12) that

$$(13) \quad \begin{aligned} \mathcal{E}(u) &= \|Mu\|^2 + \frac{2}{3} \|D_1 D_0 u\|^2 \\ &\quad + \left\| \sqrt{\frac{2}{3}} x_1 \xi_n D_0 u + b_0 \sqrt{\frac{3}{2}} x_1^2 \xi_n^2 u \right\|^2 + \frac{2}{9} \|D_1^2 u\|^2 \\ &\quad + \frac{2}{9} \left(1 - \frac{27}{4} b_0^2 \right) \|x_1^2 \xi_n^2 u\|^2 + \frac{4}{9} \|x_1 D_1 u\|^2 - \frac{4}{9} \xi_n^2 \|u\|^2. \end{aligned}$$

Multiplying by W and integrating from 0 to ∞ we have

$$(14) \quad \begin{aligned} &\int_0^\infty 2W \operatorname{Im} \langle Pu, Mu \rangle dx_0 \\ &= W(0) \mathcal{E}(u)(0) + 2\tau \langle \xi_n \rangle^{\frac{1}{s}} \int_0^\infty W \left\{ \|Mu\|^2 + \frac{2}{3} \|D_1 D_0 u\|^2 \right. \\ &\quad + \left\| \sqrt{\frac{2}{3}} x_1 \xi_n D_0 u + b_0 \sqrt{\frac{3}{2}} x_1^2 \xi_n^2 u \right\|^2 + \frac{2}{9} \|D_1^2 u\|^2 \\ &\quad + \frac{2}{9} \left(1 - \frac{27}{4} b_0^2 \right) \|x_1^2 \xi_n^2 u\|^2 + \frac{4}{9} \|x_1 D_1 u\|^2 - \frac{4}{9} \xi_n^2 \|u\|^2 \Big\} dx_0 \\ &\quad - 2b_0 \xi_n^3 \int_0^\infty W \operatorname{Re} \langle x_1^2 u, D_1 u \rangle dx_0. \end{aligned}$$

Recalling (3) from Lemma (2.1) now with $1 \leq s \leq 2$ we can dispose of the negative contribution in (14) $-\frac{4}{9}\xi_n^2\|u\|^2$.

Let us now deal with the remainder term

$$\mathcal{R}(u) = -2b_0\xi_n^3 \int_0^\infty W \operatorname{Re}\langle x_1^2 u, D_1 u \rangle dx_0.$$

Applying Cauchy-Schwarz inequality twice we get

$$\begin{aligned} |2 \operatorname{Re}\langle x_1^2 \xi_n^2 u, \xi_n^1 \langle \xi_n \rangle^{-\frac{1}{s}} D_1 u \rangle| &\leq \|x_1^2 \xi_n^2 u\|^2 + \langle \xi_n \rangle^{2-\frac{2}{s}} \|D_1 u\|^2 \\ (15) \qquad \qquad \qquad &= \|x_1^2 \xi_n^2 u\|^2 + \langle \xi_n \rangle^{2-\frac{2}{s}} \langle D_1^2 u, u \rangle \\ &\leq \|x_1^2 \xi_n^2 u\|^2 + \langle \xi_n \rangle^{4-\frac{4}{s}} \|u\|^2 + \|D_1^2 u\|^2. \end{aligned}$$

It is clear that $\|D_0^2 u\| \leq 4(\|Mu\|^2 + \|x_1^2 \xi_n^2 u\|^2 + \|D_1^2 u\|^2)$. Using (14) and $1 + \frac{2}{s} \geq 2 \geq 4 - \frac{4}{s}$ we obtain for any $u \in C_0^\infty(\mathbb{R}^n)$

$$\begin{aligned} \int_0^\infty W \|Pu\|^2 dx_0 &\geq CW(0) \sum_{j=0}^2 \tau^{4-3j/2} \langle \xi_n \rangle^{\frac{5-2j}{s}} E_j(u(0)) \\ (16) \qquad \qquad \qquad &+ C \sum_{j=0}^2 \tau^{6-2j} \int_0^\infty W \langle \xi_n \rangle^{\frac{6-2j}{s}} E_j(u(x_0)) dx_0 \end{aligned}$$

if τ is large enough and $1 \leq s \leq 2$. If we choose $s = 3/2$ so that we have $\langle \xi_n \rangle^4 E_0(u(x_0)) = E_0(\langle \xi_n \rangle^2 u(x_0))$ which control any lower order term and we arrive at the Bronshtein's theorem (see [4]).

Let $s = 2$ and

$$\tilde{E}_j(u(x_0)) = \|D_0^j u(x_0)\|^2 + \|D_1^j u(x_0)\|^2 + \|(x_1 D_n)^j u(x_0)\|^2.$$

Then for any $u \in C_0^\infty(\mathbb{R}^{n+1})$ vanishing in $x_0 \geq a$ we integrate (3) with respect to ξ_n we get

$$\begin{aligned} \int_0^a \|e^{\tau \langle D_n \rangle^{\frac{1}{2}}(x_0-a)} Pu\|^2 dx_0 &\geq C \sum_{j=0}^2 \tilde{E}_j(e^{-\tau a \langle D_n \rangle^{\frac{1}{2}}} \langle D_n \rangle^{\frac{5-2j}{4}} u(0)) \\ (17) \qquad \qquad \qquad &+ C \int_0^a \sum_{j=0}^2 \tilde{E}_j(e^{\tau \langle D_n \rangle^{\frac{1}{2}}(x_0-a)} \langle D_n \rangle^{\frac{6-2j}{4}} u(x_0)). \end{aligned}$$

Let us denote $\langle \xi \rangle = \sqrt{1 + \sum_{j=1}^n \xi_j^2}$. Note that

$$\langle D \rangle^s x_1^k = \sum_{\ell=0}^k \frac{1}{\ell!} x_1^{k-\ell} \phi_{s\ell}(D), \quad \phi_{s\ell}(\xi) = (-i)^\ell \frac{\partial^\ell}{\partial \xi_1^\ell} \langle \xi \rangle^s.$$

Then writing $\langle D \rangle^s Pu = (P + R)\langle D \rangle^s u$ it is easy to check

Lemma 2.2. *For any $s \in \mathbb{R}$ there exist $C = C_s > 0$, $\tau = \tau_s > 0$ such that for any $u \in C_0^\infty(\mathbb{R}^{n+1})$ vanishing in $x_0 \geq a$*

$$\begin{aligned}
 (18) \quad & \int_0^a \|e^{\tau \langle D_n \rangle^{\frac{1}{2}}(x_0-a)} \langle D \rangle^s P u\|^2 dx_0 \\
 & \geq C \sum_{j=0}^2 \tilde{E}_j \left(e^{-\tau a \langle D_n \rangle^{\frac{1}{2}}} \langle D_n \rangle^{\frac{5-2j}{4}} \langle D \rangle^s u(0) \right) \\
 & + C \int_0^a \sum_{j=0}^2 \tilde{E}_j \left(e^{\tau \langle D_n \rangle^{\frac{1}{2}}(x_0-a)} \langle D_n \rangle^{\frac{6-2j}{4}} \langle D \rangle^s u(x_0) \right) dx_0.
 \end{aligned}$$

Let $E = \{Pv \mid v \in C_0^\infty(\mathbb{R}^{n+1} \cap \{x_0 < a\})\}$ and let $s > 0$ large. Consider the anti-linear functional

$$\Phi : Pv \mapsto \sum_{j=0}^2 (\phi_{2-j}, D_0^j v(0)) + \int_0^a (f, v) dx_0$$

where we assume that

$$\begin{aligned}
 (19) \quad & e^{\tau a \langle D_n \rangle^{\frac{1}{2}}} \langle D_n \rangle^{\frac{-(5-2j)}{4}} \langle D \rangle^s \phi_{2-j} \in L^2(\mathbb{R}^n), \\
 & e^{-\tau \langle D_n \rangle^{\frac{1}{2}}(x_0-a)} \langle D_n \rangle^{-\frac{3}{2}} \langle D \rangle^s f \in L^2((0, a) \times \mathbb{R}^n).
 \end{aligned}$$

From Lemma 2.2 we have

$$\begin{aligned}
 & \sum_{j=0}^2 |(\phi_{2-j}, D_0^j v(0))| + \left| \int_0^a (f, v) dx_0 \right| \\
 & \leq \left(\sum_{j=0}^2 \|e^{\tau a \langle D_n \rangle^{\frac{1}{2}}} \langle D_n \rangle^{\frac{-(5-2j)}{4}} \langle D \rangle^s \phi_{2-j}\|^2 \right)^{1/2} \\
 & \times \left(\sum_{j=0}^2 \|e^{-\tau a \langle D_n \rangle^{\frac{1}{2}}} \langle D_n \rangle^{\frac{5-2j}{4}} \langle D \rangle^{-s} D_0^j v(0)\|^2 \right)^{1/2} \\
 & + \left(\int_0^a \|e^{-\tau \langle D_n \rangle^{\frac{1}{2}}(x_0-a)} \langle D_n \rangle^{-\frac{3}{2}} \langle D \rangle^s f\|^2 dx_0 \right)^{1/2} \\
 & \times \left(\int_0^a \|e^{\tau \langle D_n \rangle^{\frac{1}{2}}(x_0-a)} \langle D_n \rangle^{\frac{3}{2}} \langle D \rangle^{-s} v\|^2 dx_0 \right)^{1/2} \\
 & \leq C \left(\int_0^a \|e^{\tau \langle D_n \rangle^{\frac{1}{2}}(x_0-a)} \langle D \rangle^{-s} Pv\|^2 dx_0 \right)^{1/2}.
 \end{aligned}$$

From the Hahn-Banach theorem Φ can be extended to a bounded linear functional on $\{u \mid e^{\tau \langle D_n \rangle^{\frac{1}{2}}(x_0-a)} \langle D \rangle^{-s} u \in L^2((0, a) \times \mathbb{R}^n)\}$. Then there

exists u such that

$$\int_0^a \|e^{-\tau\langle D_n \rangle^{\frac{1}{2}}(x_0-a)} \langle D \rangle^s u\|^2 dx_0 < +\infty$$

and satisfies

$$T(g) = \int_0^a (u, g) dx_0.$$

When $g = Pv$ one has

$$\sum_{j=0}^2 (\phi_j, D_0^j v(0)) + \int_0^a (f, v) dx_0 = \int_0^a (u, Pv) dx_0$$

for any $v \in C_0^\infty(\mathbb{R}^{n+1})$ with $v = 0$ for $x_0 \geq a$. Choosing v so that $v \in C_0^\infty(\{0 < x_0 < a\})$ we see that $Pu = f$ in $(0, a) \times \mathbb{R}^n$ since $P^* = P$. Thus we have

$$\begin{aligned} \sum_{j=0}^2 (\phi_{2-j}, D_0^j v(0)) &= -i(u(0), D_0^2 v(0)) \\ &\quad -i(D_0 u(0), D_0 v(0)) - i(D_0^2 u(0), v(0)) + i(\Omega u(0), v(0)). \end{aligned}$$

From this it follows that

$$u(0) = i\phi_0, \quad D_0 u(0) = i\phi_1, \quad D_0^2 u(0) = i\phi_2 + \Omega\phi_0.$$

Since $e^{-\tau\langle D_n \rangle^{\frac{1}{2}}(x_0-a)} \langle D \rangle^s u = U \in L^2((0, a) \times \mathbb{R}^n)$ we have

$$u = e^{\tau\langle D_n \rangle^{\frac{1}{2}}(x_0-a)} \langle D \rangle^{-s} U$$

hence it is clear that $u \in L^2((0, a); H^s(\mathbb{R}^n))$. Since we have $Pu \in L^2((0, a); H^{s-3/2}(\mathbb{R}^n))$ from the assumption then from Theorem B.2.9 ([6], vol.3) it follows that

$$D_0^j u \in L^2((0, a); H^{s-3/2-j}(\mathbb{R}^n))$$

for $j = 0, 1, 2, 3$. Thus we get a smooth solution in $(0, a) \times \mathbb{R}^n$ provided (19) is verified and choosing s large. □

3. OPTIMALITY OF THE GEVREY INDEX

3.1. Sibuya's results. The differential equation

$$(20) \quad w''(y) = (y^3 + \zeta y)w(y)$$

will play a very important role in the construction of the family of solutions leading to the optimality of the Gevrey index $s = 2$.

Therefore we recap briefly, in this special setting, the general theory of subdominant solutions and Stokes coefficients for the equation (20), following the presentation found, for example, in the book of Sibuya [15].

Theorem 6.1 in [15] states that the differential equation (20) has a *unique* solution

$$w(y; \zeta) = \mathcal{Y}(y; \zeta)$$

such that

- (i) $\mathcal{Y}(y; \zeta)$ is an entire function of (y, ζ) .
- (ii) $\mathcal{Y}(y; \zeta)$ and its derivative $\mathcal{Y}'(y; \zeta)$ admit an asymptotic representation

$$(21) \quad \mathcal{Y}(y; \zeta) \sim y^{-3/4} \left[1 + \sum_{N=1}^{\infty} B_N y^{-N/2} \right] \exp[-E(y; \zeta)]$$

$$(22) \quad \mathcal{Y}'(y; \zeta) \sim y^{3/4} \left[-1 + \sum_{N=1}^{\infty} C_N y^{-N/2} \right] \exp[-E(y; \zeta)]$$

uniformly on each compact set in the ζ space as y goes to infinity in any closed subsector of the open sector

$$|\arg y| < \frac{3\pi}{5};$$

moreover

$$E(y; \zeta) = \frac{2}{5}y^{5/2} + \zeta y^{1/2}$$

and B_N, C_N are polynomials in ζ .

We note that if we set $\omega = \exp[i\frac{2\pi}{5}]$ and

$$\mathcal{Y}_k(y; \zeta) = \mathcal{Y}(\omega^{-k}y; \omega^{-2k}\zeta)$$

where $k = 0, 1, 2, 3, 4$ then all the five functions $\mathcal{Y}_k(y; \zeta)$ solve (20). In particular $\mathcal{Y}_0(y; \zeta) = \mathcal{Y}(y; \zeta)$. Let us denote

$$Y = y^{-3/4} \left[1 + \sum_{N=1}^{\infty} B_N y^{-N/2} \right] \exp[-E(y; \zeta)]$$

then we have

- (i) $\mathcal{Y}_k(y; \zeta)$ is an entire function of (y, ζ) .
- (ii) $\mathcal{Y}_k(y; \zeta) \sim Y(\omega^{-k}y; \omega^{-2k}\zeta)$ uniformly on each compact set in the ζ space as y goes to infinity in any closed subsector of the open sector

$$|\arg y - \frac{2k}{5}\pi| < \frac{3\pi}{5}.$$

Let S_k denote the open sector defined by $|\arg y - \frac{2k}{5}\pi| < \frac{\pi}{5}$. We say that a solution of (20) is subdominant in the sector S_k if it tends to 0 as y tends to infinity along any direction in the sector S_k . Analogously a solution is

called dominant in the sector S_k if this solution tends to ∞ as y tends to infinity along any direction in the sector S_k .

Since

$$(23) \quad \operatorname{Re}[y^{5/2}] > 0 \quad \text{for } y \in S_0$$

and $\operatorname{Re}[y^{5/2}] < 0$ for $y \in S_{-1} = S_4$ and for S_1 the solution $\mathcal{Y}_0(y; \zeta)$ is subdominant in S_0 and dominant in S_4 and S_1 . Similarly $\mathcal{Y}_k(y; \zeta)$ is subdominant in S_k and dominant in S_{k-1} and S_{k+1} . It is clear that \mathcal{Y}_{k+1} and \mathcal{Y}_{k+2} are linearly independent. Therefore \mathcal{Y}_k is a linear combination of those two:

$$\mathcal{Y}_k(y; \zeta) = C_k(\zeta)\mathcal{Y}_{k+1}(y; \zeta) + \tilde{C}_k(\zeta)\mathcal{Y}_{k+2}(y; \zeta).$$

The above relation, connection formula for $\mathcal{Y}_k(y; \zeta)$ and the coefficients C_k , \tilde{C}_k are called the Stokes coefficients for $\mathcal{Y}_k(y; \zeta)$. We summarize in the following statement some of the known and useful facts about the Stokes coefficients for our particular equation (20). Proofs can be found in Chapter 5 of [15].

Proposition 3.1. *The following results hold.*

- (i) $\tilde{C}_k(\zeta) = -\omega$, $\forall k$, and ζ ,
- (ii) $C_k(\zeta) = C_0(\omega^{-2k}\zeta)$, $\forall k$, ζ and $C_0(\zeta)$ is an entire function of ζ ,
- (iii) For each fixed ζ there exists k such that $C_k(\zeta) \neq 0$,
- (iv) $C_k(0) = 1 + \omega$, $\forall k$,
- (v) $\partial_\zeta C_0(\zeta)|_{\zeta=0} \neq 0$.

We also have

Proposition 3.2. *If we set*

$$S_k(\zeta) = \begin{bmatrix} C_k(\zeta) & 1 \\ -\omega & 0 \end{bmatrix}, \quad k = 0, 1, 2, 3, 4$$

then we have

$$(24) \quad S_4(\zeta) \cdot S_3(\zeta) \cdot S_2(\zeta) \cdot S_1(\zeta) \cdot S_0(\zeta) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

The proof of Proposition 3.2 is straightforward. Applying this proposition we have an interesting result.

Proposition 3.3. *(24) is equivalent to*

$$C_k(\zeta) + \omega^2 C_{k+2}(\zeta) C_{k+3}(\zeta) - \omega^3 = 0 \quad \text{mod } 5.$$

Or otherwise stated

$$C_0(\zeta) + \omega^2 C_0(\omega\zeta) C_0(\omega^4\zeta) - \omega^3 = 0, \quad \forall \zeta \in \mathbb{C}.$$

Proof: A straightforward computation from (24).

We now state a key lemma which is proved in [2]. We repeat here the short proof.

Lemma 3.1. *The Stokes coefficient $C_0(\zeta)$ vanishes in at least one (non zero) ζ_0 .*

Proof: Suppose that $C_0(\zeta) \neq 0$ for all $\zeta \in \mathbb{C}$. Then from Proposition 3.3 that it follows that $C_0(\zeta) \neq \omega^3$ for all $\zeta \in \mathbb{C}$. Since $C_0(\zeta)$ is an entire function Picard's Little Theorem implies that $C_0(\zeta)$ would be constant because $C_0(\zeta)$ avoids two distinct values 0 and ω^3 . But this contradicts (v) of Proposition 3.1. Thus there exists ζ_0 with $C_0(\zeta_0) = 0$ where the fact $\zeta_0 \neq 0$ follows from (iv).

3.2. Localization of zeros. Now we know that $C_0(\zeta)$ vanishes somewhere, we would like to find out where exactly this happens. We are going to begin with a symmetry result:

Lemma 3.2. *The Stokes coefficient $C_0(\zeta)$ verifies the equivalence*

$$(25) \quad C_0(\zeta) = 0 \iff C_0(\overline{\omega\zeta}) = 0.$$

Proof. We see how $\overline{\mathcal{Y}_0(\overline{y}; \overline{\zeta})}$ is a solution of (20) whose asymptotic behavior in the sector S_0 is the same as that of $\mathcal{Y}_0(y; \zeta)$. The uniqueness of the canonical Sibuya solution implies thus that

$$\overline{\mathcal{Y}_0(y; \zeta)} = \mathcal{Y}_0(\overline{y}; \overline{\zeta})$$

Recall that $\mathcal{Y}_k(y; \zeta) = \mathcal{Y}(\omega^{-k}y; \omega^{-2k}\zeta)$ and that

$$\mathcal{Y}_k(y; \zeta) = C_k(\zeta)\mathcal{Y}_{k+1}(y; \zeta) - \omega\mathcal{Y}_{k+2}(y; \zeta).$$

It is easy to verify that $\overline{\mathcal{Y}_4(y; \zeta)} = \mathcal{Y}_1(\overline{y}; \overline{\zeta})$ and that $\overline{\mathcal{Y}_1(y; \zeta)} = \mathcal{Y}_4(\overline{y}; \overline{\zeta})$. We conjugate

$$(26) \quad \mathcal{Y}_4(y; \zeta) = C_4(\zeta)\mathcal{Y}_0(y; \zeta) - \omega\mathcal{Y}_1(y; \zeta)$$

and have

$$(27) \quad \mathcal{Y}_1(\overline{y}; \overline{\zeta}) = \overline{C_4(\zeta)}\mathcal{Y}_0(\overline{y}; \overline{\zeta}) - \overline{\omega}\mathcal{Y}_4(\overline{y}; \overline{\zeta}).$$

Switch to \overline{y} and $\overline{\zeta}$ in (26) and we get

$$(28) \quad \mathcal{Y}_4(\overline{y}; \overline{\zeta}) = C_4(\overline{\zeta})\mathcal{Y}_0(\overline{y}; \overline{\zeta}) - \omega\mathcal{Y}_1(\overline{y}; \overline{\zeta}).$$

Multiplying (27) by $\omega^{-3/4}$ and (28) by $\omega^{3/4}$ we get:

$$\omega^{-3/4}\mathcal{Y}_1(\overline{y}; \overline{\zeta}) = \overline{\omega^{3/4}C_4(\zeta)}\mathcal{Y}_0(\overline{y}; \overline{\zeta}) + \omega^{3/4}\mathcal{Y}_4(\overline{y}; \overline{\zeta}),$$

$$\omega^{3/4}\mathcal{Y}_4(\bar{y}; \bar{\zeta}) = \omega^{3/4}C_4(\bar{\zeta})\mathcal{Y}_0(\bar{y}; \bar{\zeta}) + \omega^{-3/4}\mathcal{Y}_1(\bar{y}; \bar{\zeta}).$$

Adding these two equations we have:

$$\left(\overline{\omega^{3/4}C_4(\zeta)} + \omega^{3/4}C_4(\bar{\zeta})\right)\mathcal{Y}_0(\bar{y}; \bar{\zeta}) = 0,$$

from which we have

$$\overline{C_4(\zeta)} = 0 \iff C_4(\bar{\zeta}) = 0,$$

that is

$$\overline{C_0(\omega^2\zeta)} = 0 \iff C_0(\omega^2\bar{\zeta}) = 0,$$

therefore

$$\overline{C_0(\zeta)} = 0 \iff C_0(\omega^4\bar{\zeta}) = C_0(\bar{\omega\zeta}) = 0,$$

this last equality proving the Lemma. \square

The following is a very important step in the construction of the null solutions, and is the sharpest result, at least to the authors' knowledge, on the location of the zeros of the entire function $C_0(\zeta)$.

Lemma 3.3. *There exists $\zeta_0 \in S = \{z \in \mathbb{C} | \pi < \arg z \leq \frac{19}{15}\pi\}$ where $C_0(\zeta_0) = 0$.*

Proof. We recall from Proposition 3.1 in [13] that $C_0(\zeta) = 0$ implies either $\zeta \in S_1 = \{\pi \leq \arg \zeta \leq \frac{19}{15}\pi\}$ or $\zeta \in S_2 = \{\frac{\pi}{3} \leq \arg \zeta \leq \frac{3}{5}\pi\}$. But S_1 and S_2 are symmetric under the mapping $\zeta \rightarrow \omega\bar{\zeta}$. We just have to show the $\arg \zeta \neq \pi$. Proposition (3.3) and Lemma (3.2) above together imply that $C_0(\zeta) \neq 0$ if ζ is real. \square

3.3. Proof of Theorem (1.1). Consider again the operator:

$$(29) \quad P_3(x, D) = D_0^3 - (D_1^2 + x_1^2 D_n^2)D_0 - b_0 x_1^3 D_n^3.$$

In the following we will choose $b_0 = \frac{\sqrt{2}}{3\sqrt{3}}$, which clearly satisfies the hyperbolicity assumption $b_0^2 \leq 4/27$.

Let $\lambda > 0$ a positive large parameter, $R > 0, \theta \in]0, \pi[$ to be chosen later and consider:

$$(30) \quad U(x, \lambda, R, \theta) = e^{ix_0\lambda^{\frac{1}{2}}Re^{i\theta} + ix_n\lambda}u(x_1, \lambda, R, \theta)$$

Here $x = (x_0, x_1, x'', x_n)$, sometimes the x'' components will be omitted to enhance readability.

From (30) let us set

$$U(x, \lambda, R, \theta) = E(x_0, x_n, \lambda) \times w(Ax_1 + B),$$

with $E(x_0, x_n, \lambda) = e^{ix_0\lambda^{\frac{1}{2}}Re^{i\theta} + ix_n\lambda}$ and A, B to be chosen together with w . It is clear that $D_0U = \lambda^{1/2}Re^{i\theta}U$, $D_nU = \lambda U$ and $D_1U = -iEAw'(Ax_1 + B)$.

Therefore we have

$$(31) \quad PU = U \left(\lambda^{3/2}R^3e^{i3\theta} - \lambda^{5/2}Re^{i\theta}x_1^2 - b_0\lambda^3x_1^3 + \lambda^{1/2}Re^{i\theta}A^2\frac{w''}{w}(Ax_1 + B) \right).$$

Thus setting $y = Ax_1 + B$ we have from (31) and the request that $PU = 0$,

$$(32) \quad \begin{aligned} w''(y) = & \lambda^{-1/2}R^{-1}e^{-i\theta}A^{-2} \left[\frac{b_0\lambda^3}{A^3}y^3 + \left(-3\frac{b_0\lambda^3B}{A^3} + \frac{\lambda^{5/2}Re^{i\theta}}{A^2} \right) y^2 \right. \\ & \left. + \left(\frac{3b_0\lambda^3B^2}{A^3} - \frac{2\lambda^{5/2}Re^{i\theta}B}{A^2} \right) y - \frac{b_0\lambda^3B^3}{A^3} + \frac{\lambda^{5/2}Re^{i\theta}B^2}{A^2} - \lambda^{3/2}e^{i3\theta}R^3 \right] w(y). \end{aligned}$$

The following choices are then made:

$$(33) \quad \lambda^{-1/2}R^{-1}e^{-i\theta}\frac{b_0\lambda^3}{A^5} = 1,$$

and

$$(34) \quad -\frac{3b_0\lambda^3B}{A^3} + \frac{\lambda^{5/2}Re^{i\theta}}{A^2} = 0.$$

(33) and (34) yield

$$(35) \quad A = \lambda^{1/2}b_0^{1/5}R^{-1/5}e^{-i\theta/5}, \quad B = \frac{R^{4/5}b_0^{-4/5}e^{i4\theta/5}}{3}.$$

Using these values we have from (32)

$$(36) \quad w''(y) = (y^3 + \zeta y + \mu)w(y),$$

with

$$(37) \quad \zeta = -\frac{b_0^{-8/5}e^{i8\theta/5}R^{8/5}}{3}, \quad \mu = \lambda R^2e^{2i\theta}A^{-2} \left(\frac{2}{27b_0^2} - 1 \right).$$

It is now clear that choosing $b_0 = \frac{\sqrt{2}}{3\sqrt{3}}$ will give us equation (20).

If we do not impose this last condition we would be left with the more difficult task of finding θ and R such that

$$C_0\left(-\frac{b_0^{-8/5}e^{i8\theta/5}R^{8/5}}{3}, R^{12/5}e^{12i\theta/5}b_0^{-2/5}\left(\frac{2}{27b_0^2}-1\right)\right) = 0.$$

We now choose $w(y; \zeta) = \mathcal{Y}_0(y; \zeta_0)$ with ζ_0 found in Lemma 3.3 and from (35) we take $y = b_0^{\frac{1}{5}}R^{-\frac{1}{5}}\lambda^{\frac{1}{2}}e^{-i\frac{\theta}{5}}x_1 + \frac{1}{3}b_0^{-\frac{4}{5}}R^{\frac{4}{5}}e^{\frac{4i\theta}{5}}$.

We have that $\frac{1}{3}b_0^{-\frac{8}{5}}R^{\frac{8}{5}}e^{i\frac{8\theta}{5}+i\pi} = |\zeta_0|e^{i\arg \zeta_0}$ and $\pi < \arg \zeta_0 \leq \frac{19}{15}\pi$.

This clearly leaves us with $0 < \theta_0 = \theta(\arg \zeta_0) \leq \frac{\pi}{6}$, while the number R , still at our disposal, is chosen to fix the absolute values, thus $R = R_0 > 0$, depending on b_0 and $|\zeta_0|$.

Recall that $\mathcal{Y}_k(y; \zeta) = \mathcal{Y}(\omega^{-k}y; \omega^{-2k}\zeta)$ and that

$$(38) \quad \mathcal{Y}_0(y; \zeta_0) = -\omega\mathcal{Y}_2(y; \zeta_0) = -\omega\mathcal{Y}_0(\omega^{-2}y; \omega^{-4}\zeta_0),$$

since $C_0(\zeta_0) = 0$.

Thus we notice that when $x_1 > 0$ and λ is large $\arg(y) \in [-\frac{\pi}{30}, 0[$ clearly well inside the subdominant sector S_0 .

On the other hand if $x_1 < 0$ and λ is large, using (38), we have that $\arg(y) \in [\frac{\pi}{6}, \frac{\pi}{5}[$, again within the subdominant sector S_0 .

This proves in particular that $u(x_1, \lambda, R_0, \theta_0)$ is, for every $\lambda > 0$ in the Schwartz space $\mathcal{S}(\mathbb{R})$ and moreover $u(x_1, \lambda, R_0, \theta_0)$ is bounded on \mathbb{R} uniformly in λ .

Let

$$U_\lambda = e^{i(T-x_0)\lambda^{1/2}Re^{i\theta}-ix_n\lambda}w(y, \zeta_0)$$

then $PU_\lambda = 0$ because $P(x_1, -D_0, D_1, -D_n) = -P(x_1, D_0, D_1, D_n)$. Let u be a solution to the Cauchy problem

$$\begin{cases} Pu = 0, \\ u(0, x') = 0, \quad D_0u(0, x') = 0, \quad D_0^2u(0, x') = \bar{\phi}(x_1)\bar{\psi}(x'')\bar{\theta}(x_n) \end{cases}$$

where $\phi \in C_0^\infty(\mathbb{R})$, $\psi \in C_0^\infty(\mathbb{R}^{n-2})$, $\theta \in C_0^\infty(\mathbb{R})$. Let us set

$$D_\delta = \{x \in \mathbb{R}^{n+1} \mid |x'|^2 + |x_0| < \delta\}$$

and recall the Holmgren theorem (see for example [11] Theorem 4.2):

Proposition 3.4. *There exists $\epsilon_0 > 0$ such that; let $0 < \epsilon < \epsilon_0$ and $u(x) \in C^2(D_\epsilon)$ verifies*

$$\begin{cases} Pu = 0 & \text{in } D_\epsilon \\ D_0^j u(0, x') = 0, \quad j = 0, 1, & x \in D_\epsilon \cap \{x_0 = 0\} \end{cases}$$

then $u(x)$ vanishes identically in D_ϵ .

From this proposition we can assume that

$$u(x) = 0$$

if $0 \leq x_0 \leq T$, $|x'| \geq r$ for small $T > 0$, $r > 0$. Then

$$\begin{aligned} 0 &= \int_0^T (PU_\lambda, u) dx_0 = \int_0^T (U_\lambda, Pu) dx_0 - i(D_0^2 U_\lambda(T), u(T)) \\ &\quad - i(D_0 U_\lambda(T), D_0 u(T)) - i(U_\lambda(T), D_0^2(T)) + i(U_\lambda(0), D_0^2 u(0)) \\ &\quad + i((D_1^2 + x_1^2 D_n^2) U_\lambda(T), u(T)). \end{aligned}$$

Hence

$$\begin{aligned} (U_\lambda(0), D_0^2 u(0)) &= (D_0^2 U_\lambda(T), u(T)) \\ &\quad + (D_0 U_\lambda(T), D_0 u(T)) + (U_\lambda(T), D_0^2(T)) \\ &\quad - ((D_1^2 + x_1^2 D_n^2) U_\lambda(T), u(T)). \end{aligned}$$

The right-hand side is $O(\lambda^2)$ because $w(y, \zeta_0)$, $\lambda^{-1/2} D_1 w(y, \zeta_0)$ are bounded uniformly in λ . On the other hand the left-hand side is

$$\begin{aligned} &\hat{\theta}(\lambda) e^{iT\lambda^{1/2} Re^{i\theta}} \int w(y, \zeta_0) \phi(x_1) \psi(x'') dx_1 dx'' \\ &= \hat{\theta}(\lambda) e^{iT\lambda^{1/2} R^{1/2} e^{i\theta}} \left(\psi(x'') dx'' \right) \int w(y, \zeta_0) \phi(x_1) dx_1. \end{aligned}$$

We choose ψ so that $\int \psi(x'') dx'' \neq 0$. Recall that $\theta \in \gamma_0^{(2)}(\mathbb{R})$ if and only if we have

$$|\hat{\theta}(\xi)| \leq C e^{-L|\xi|^{1/2}}$$

with some $L > 0$, $C > 0$. Thus if we take $\theta \notin \gamma_0^{(2)}(\mathbb{R})$ which is even then $\rho^{-N} \hat{\theta}(\lambda) e^{iT\lambda^{1/2} Re^{i\theta}}$ is not bounded as $\lambda \rightarrow \infty$. We must check that

$$\int w(y, \zeta_0) \phi(x_1) dx_1 \rightarrow c \neq 0$$

with a suitable choice of ϕ . Let $\alpha = b_0^{1/5} R^{-1/5} e^{-i\theta/5}$, $\beta = b_0^{-4/5} R^{4/5} e^{4i\theta/5}/3$. We have

$$\begin{aligned} \int w(\lambda^{1/2} \alpha x_1 + \beta, \zeta_0) \phi(x_1) dx_1 &= \lambda^{-1/2} \int w(\alpha x_1 + \beta, \zeta_0) \phi(\lambda^{-1/2} x_1) dx_1 \\ &= \lambda^{-1/2} \left[\sum_{k=0}^2 \frac{\lambda^{-k/2}}{k!} \phi^{(k)}(0) \int w(\alpha x_1 + \beta, \zeta_0) x_1^k dx_1 + O(\lambda^{-3/2}) \right]. \end{aligned}$$

It is enough to show that we have $\int w(\alpha x_1 + \beta, \zeta_0) x_1^k dx_1 \neq 0$ for at least one $k = 0, 1, 2$. Put

$$v(\xi) = \int e^{-ix\xi} w(\alpha x + \beta, \zeta_0) dx$$

Then $v(\xi)$ satisfies the equation

$$\left(i\alpha\frac{d}{d\xi} + \beta\right)^3 v(\xi) + \zeta_0 \left(i\alpha\frac{d}{d\xi} + \beta\right) v(\xi) + \alpha^{-2} \xi^2 v(\xi) = 0$$

and

$$v^{(k)}(0) = \int (-ix)^k w(\alpha x + \beta, \zeta_0) dx = (-i)^k \int w(\alpha x + \beta, \zeta_0) x^k dx.$$

So if $v^{(k)}(0) = 0$ for $k = 0, 1, 2$ then we would have $v(\xi) = 0$ so that $w(\alpha x + \beta, \zeta_0) = 0$ which is a contradiction.

4. CONES AND FACTORIZATION

Here we briefly verify that the propagation cone is not transversal to the triple manifold.

Let $p(x, \xi) = \xi_0^3 - (\xi_1^2 + x_1^2 \xi_n^2) \xi_0 - b_0 x_1^3 \xi_n^3$ be the principal symbol of the operator (1). p vanishes exactly of order 3 on $\Sigma_3 = \{x_1 = \xi_0 = \xi_1 = 0\}$ near $(0; 0, \dots, 1)$ if $|b_0| < \frac{2}{3\sqrt{3}}$. Fix $z \in \Sigma_3$ and take $\delta v = (-1, 0, \dots, 0; 0)$. Clearly $\delta v \in T_z \Sigma_3$ and, since $\sigma(\delta v, (\delta y, \delta \eta)) = -\delta \eta_0 \leq 0$ if $(\delta y, \delta \eta) \in \Gamma_z$, we have that $C_z \cap T_z \Sigma_3 \neq \emptyset$. On the other hand C_z cannot be completely contained in $T_z \Sigma_3$, because otherwise $T_z \Sigma_3 \subset C_z^\sigma$ and this would imply that $\langle H_{\xi_0}, H_{\xi_1}, H_{x_1} \rangle \subset \overline{\Gamma_z}$, which is false. Therefore C_z is neither disjoint from nor totally inside $T_z \Sigma_3$.

For the next item we change slightly the notations in order to simplify the treatment of a third degree equation naturally associated with the problem. Let us show that for our model no root is C^∞ .

Let $p = \tau^3 - 3(x^2 + \xi^2)\tau - 2bx^3$, with $0 < |b| < 1$. If p could be written like

$$p = (\tau - L(x, \xi))(\tau^2 + A(x, \xi)\tau + B(x, \xi))$$

with L, A, B regular C^∞ functions, one then would get $A = L$, $L^2 - B = 3(x^2 + \xi^2)$ and $LB = 2bx^3$.

This shows that at $x = 0$ there should always exist a regular root $\tau(0, \xi) = 0$ identically.

The discriminant is $\Delta = 108\{(x^2 + \xi^2)^3 - b^2 x^6\}$. Putting $p = -3(x^2 + \xi^2)$, $q = -2bx^3$ we have

$$-\frac{q}{2} = \rho \cos \phi, \quad \sqrt{\frac{\Delta}{108}} = \rho \sin \phi$$

with

$$\rho = \sqrt{\left(-\frac{p}{3}\right)^3} = (x^2 + \xi^2)^{3/2}, \quad \cos \phi = -\frac{q}{2\rho} = \frac{bx^3}{(x^2 + \xi^2)^{3/2}}.$$

Thus we have

$$\phi(x, \xi) = \arccos\left(\frac{bx^3}{(x^2 + \xi^2)^{3/2}}\right).$$

The root vanishing identically when $x = 0$ is

$$\tau(x, \xi) = 2(x^2 + \xi^2)^{1/2} \cos\left(\frac{\arccos\left(\frac{bx^3}{(x^2 + \xi^2)^{3/2}}\right) + 4\pi}{3}\right).$$

We have

$$\arccos\left(\frac{bx^3}{(x^2 + \xi^2)^{3/2}}\right) = \frac{\pi}{2} - f(x, \xi).$$

with $f(x, \xi) = g\left(\frac{bx^3}{(x^2 + \xi^2)^{3/2}}\right)$, and

$$g(u) = \sum_{k=0}^{\infty} \frac{(2k)!u^{2k+1}}{2^{2k}(k!)^2(2k+1)}, \quad |u| < 1.$$

Therefore we have, since $|b| < 1$,

$$\begin{aligned} \tau(x, \xi) &= 2(x^2 + \xi^2)^{1/2} \cos\left(\frac{3\pi}{2} - \frac{1}{3}g\left(\frac{bx^3}{(x^2 + \xi^2)^{3/2}}\right)\right) \\ &= -2(x^2 + \xi^2)^{1/2} \sin\left(\frac{1}{3}g\left(\frac{bx^3}{(x^2 + \xi^2)^{3/2}}\right)\right). \end{aligned}$$

This implies

$$\tau(x, \xi) \sim -\frac{2}{3} \frac{bx^3}{x^2 + \xi^2} = -\frac{2}{3} x \rho(x, \xi),$$

with $\rho(x, \xi)$, not identically zero because of $b \neq 0$ cannot be continuous at the origin: this contradiction proves that there cannot be a smooth factorization for p .

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